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# Approximation by Extreme Functions

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For $T$ a topological space and $X$ a real normed space, $Y=C(T,X)$ denotes the space of continuous and bounded functions from $T$ into $X$ endowed with the sup norm. We calculate a formula for the distance $\alpha(f)$ from $f$ in $Y$ to the set $Y^{-1}$ of functions in $Y$ which have no zeros. Namely, we prove that $\alpha(f)$ is the infimum of numbers $\delta>0$ for which the continuous function $t\mapsto f(t)/\|f(t)\|$ defined for every $t$ with $\|f(t)\|\geqslant\delta$ has a continuous extension $e$ from $T$ into the unit sphere of $X$. This permits us to get the general expression of the Aron–Lohman $\lambda$-function of $Y$ when $X$ is strictly convex. We show that any function in $Y$ has a best approximation in $\overline{Y^{-1}}$ which can be chosen to have the least possible norm. If $X$ is strictly convex and $E(Y)$ denotes the set of extreme points of the unit ball of $Y$, this fact enables us to prove that $\operatorname{dist}(f, E(Y))=\max\{1-m(f)+\alpha(f), \|f\|-1\}$ $\forall f\in Y$, where $m(f)=\inf\{\|f(t)\|: t\in T\}$. Moreover, we show that $E(Y)$ is proximinal in $Y^{-1}$ and give sufficient conditions under which $f$ in $Y\backslash Y^{-1}$ admits a best approximation in $E(Y)$. © 1999 Academic Press

## 1. INTRODUCTION AND NOTATION

Throughout this paper the letter $T$ stands for a topological space, while $X$ denotes a real normed space. $B(X)$, $S(X)$, and $E(X)$ are the closed unit

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ball of $X$, the unit sphere of $X$, and the set of extreme points of $B(X)$, respectively.

We denote by $C(T, X)$ the space of $X$-valued continuous and bounded functions on $T$ equipped with the supremum norm. To simplify the notation we will frequently write $Y$ instead of $C(T, X)$.

Moreover, $Y^{-1}$ will denote the set of the functions in $Y$ which have no zeros. That is,

$$Y^{-1} = \{ f \in C(T, X): f(t) \neq 0 \ \forall t \in T \}.$$

Let us observe that for $X = \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $Y^{-1}$ is the group of the invertible elements in the algebra $C(T, \mathbb{K})$.

For every function $f$ in $Y$, we consider the notation

$$m(f) = \inf \{ \|f(t)\|: t \in T \} \qquad \text{and} \qquad \alpha(f) = \operatorname{dist}(f, Y^{-1}).$$

In Section 2, we show that $m(f)$ is the distance from an element $f$ in $Y$ to the set $Y \backslash Y^{-1}$. Moreover, we calculate the distance $\alpha(f)$ for each $f$ in $Y$. To be more precise, we prove that $\alpha(f)$ is the infimum of numbers $\delta > 0$ for which the continuous function $t \mapsto f(t)/\|f(t)\|$ defined for every $t$ with $\|f(t)\| \geqslant \delta$, has a continuous extension $e: T \to S(X)$.

Working now in the more special case of a space $C(T, X)$ being $X$ a strictly convex normed space we see that the knowledge of the parameters $m(f)$ and $\alpha(f)$ determines the distance of $f$ to the set of extreme points of $B(Y)$. Namely, we obtain $\operatorname{dist}(f, E(Y)) = \max\{1 - m(f) + \alpha(f), \ \|f\| - 1\}$, $\forall f \in Y$.

Let $A$ be a subset in $Y$, a best approximation in $A$ for $f \in Y$ is a function $g \in A$ such that $\|f - g\| = \operatorname{dist}(f, A)$. If $A \subset B \subset Y$, the set $A$ is said to be proximinal in $B$ if every element $f$ of $B$ has a best approximation in $A$.

We show that $\overline{Y^{-1}}$ is proximinal in $Y$ with a best approximation which can be chosen to have the least possible norm (Corollary 2.4). For $Y = C(T, X)$ with $X$ a strictly convex space we study the problem of the existence of best approximations in $E(Y)$ for an element $f$ in $Y$. In fact we prove that $E(Y)$ is proximinal in $Y^{-1}$ (Corollary 2.7). Furthermore, several conditions are given that are sufficient for the existence of a best approximation in $E(Y)$ for functions in $Y \backslash Y^{-1}$ (see Proposition 2.8 and Corollaries 2.9 and 2.17).

Section 3 is devoted to the study of a geometric function, called $\lambda$-function, which was introduced in [1]. Until now, the expression of the $\lambda$-function of $C(T, X)$ is only known if $X$ is strictly convex and $C(T, X)$ has the $\lambda$-property (see Section 3 for definitions and references). The knowledge of $\alpha(f)$ for every $f$ in $C(T, X)$ permits us to get the general expression of the $\lambda$-function of this space when $X$ is strictly convex although $C(T, X)$ has not the $\lambda$-property. This new formula provides information about the problem of minimal decompositions of elements as convex combinations of extreme points.

## 2. DISTANCE TO THE EXTREME FUNCTIONS IN $C(T, X)$

PROPOSITION 2.1. *Let $T$ be a topological space and $X$ a normed space. For every $f$ in $Y$ we have $m(f) = \operatorname{dist}(f, Y \backslash Y^{-1})$.*

*Proof.* If $f \in Y \backslash Y^{-1}$, then $m(f) = \operatorname{dist}(f, Y \backslash Y^{-1}) = 0$.

Let $f$ be in $Y^{-1}$. Given $g$ in $Y \backslash Y^{-1}$, there exists some $t_0$ in $T$ such that $g(t_0) = 0$. Then clearly $m(f) \leqslant \|f(t_0)\| = \|f(t_0) - g(t_0)\| \leqslant \|f - g\|$ and so $m(f) \leqslant \operatorname{dist}(f, Y \backslash Y^{-1})$.

Conversely, given $\varepsilon > 0$, there is a $t_0 \in T$ verifying that $\|f(t_0)\| < m(f) + \varepsilon$. Define $g\colon T \to X$ by

$$g(t) = f(t) - \frac{f(t)}{\|f(t)\|} \|f(t_0)\|, \qquad \forall t \in T.$$

Clearly $g \in Y \backslash Y^{-1}$ and $\|g(t) - f(t)\| = \|f(t_0)\| < m(f) + \varepsilon$, $\forall t \in T$. Hence

$$\operatorname{dist}(f, Y \backslash Y^{-1}) \leqslant \|f - g\| \leqslant m(f) + \varepsilon$$

and, since $\varepsilon$ is arbitrary, we conclude that $\operatorname{dist}(f, Y \backslash Y^{-1}) \leqslant m(f)$ which proves the equality. ∎

To get our objectives we will need the following theorem which is, in our opinion, intrinsically interesting.

THEOREM 2.2. *Let $T$ be a topological space and $X$ a normed space. Given an element $f$ in $Y$, there exists, for every $\delta > \alpha(f)$, a continuous function $e$ from $T$ into $S(X)$ such that $e(t) = f(t)/\|f(t)\|$ for every $t \in T$ with $\|f(t)\| \geqslant \delta$.*

*Proof.* Choose $\rho$ with $\alpha(f) < \rho < \delta$. Then there is a continuous mapping $h\colon T \to X$ which has no zeros and satisfies $\|f - h\| < \rho$.

Let $\varphi\colon X \to [0, 1]$ be a continuous function such that

$$\varphi(x) = \begin{cases} 0 & \text{if } \|x\| \leqslant \rho \\ 1 & \text{if } \|x\| \geqslant \delta. \end{cases}$$

Let us define the function $g\colon T \to X$ by

$$g(t) = \varphi(f(t))\, f(t) + (1 - \varphi(f(t))\, h(t)), \qquad \forall t \in T.$$

If $\|f(t)\| \leqslant \rho$, then it is clear that $g(t) = h(t)$. If $\|f(t)\| \geqslant \delta$, we have $g(t) = f(t)$. If $\rho < \|f(t)\| < \delta$ it follows that $g(t) \neq 0$. Hence, $g$ is in $Y^{-1}$.

The proof is complete if we define $e\colon T \to X$ by $e(t) = g(t)/\|g(t)\|$ for each $t$ in $T$. ∎

In the next corollary, we obtain an useful characterization of the distance $\alpha(f)$ of an element $f$ in $Y$ to the set $Y^{-1}$.

COROLLARY 2.3. *Let $T$ be a topological space and $X$ a normed space. For every $f$ in $Y$, $\alpha(f)$ is the infimum of numbers $\delta > 0$ for which the continuous function*

$$t \mapsto \frac{f(t)}{\|f(t)\|},$$

*defined for every $t$ with $\|f(t)\| \geqslant \delta$ has a continuous extension $e\colon T \to S(X)$.*

*Proof.* From the last theorem it suffices to show that if $\delta$ is a positive real number such that there is a continuous mapping $e\colon T \to S(X)$ verifying the condition

$$e(t) = \frac{f(t)}{\|f(t)\|} \qquad \text{whenever } t \in T \text{ satisfies } \|f(t)\| \geqslant \delta,$$

then $\delta \geqslant \alpha(f)$. For it, we take $g(t) = f(t) + \delta e(t)$ for each $t \in T$. The function $g$ is in $Y^{-1}$. Indeed, if $\|f(t)\| \geqslant \delta$ we have

$$g(t) = f(t) + \delta \frac{f(t)}{\|f(t)\|} = f(t)\left(1 + \frac{\delta}{\|f(t)\|}\right) \neq 0$$

and if $\|f(t)\| < \delta$, it is clear that $\|g(t)\| = \|f(t) + \delta e(t)\| \geqslant \delta - \|f(t)\| > 0$. Moreover $\|f(t) - g(t)\| = \|\delta e(t)\| = \delta$, $\forall t \in T$. So $\alpha(f) \leqslant \|f - g\| = \delta$. ∎

Let us observe that $\alpha(f) \leqslant \|f\|$, $\forall f \in Y$. Indeed, if $x$ in $S(X)$, let $g$ be defined by $g(t) = f(t) + (\varepsilon + \|f\|)\, x$ for each $t$ in $T$ with $\varepsilon > 0$. Clearly $g \in Y^{-1}$ and $\|g - f\| = \|f\| + \varepsilon$. Hence $\alpha(f) \leqslant \|f\| + \varepsilon$ and, since $\varepsilon$ is arbitrary, we conclude that $\alpha(f) \leqslant \|f\|$.

In the corollary which follows we obtain that the distance from an element $f$ in $Y$ to the set $Y^{-1}$ is attained at some $g$ in $\overline{Y^{-1}}$, and this best approximation can be chosen to have the least possible norm.

COROLLARY 2.4. *Let $T$ be a topological space and $X$ a normed space. For each $f$ in $Y$ there is a $g$ in $\overline{Y^{-1}}$ such that $\|f - g\| = \alpha(f)$ and $\|g\| = \|f\| - \alpha(f)$. In other words, $\overline{Y^{-1}}$ is proximinal in $Y$.*

*Proof.* If $\alpha(f)=0$ or $\alpha(f)=\|f\|$, we can take $g=f$ and $g=0$, respectively, and the desired conclusion holds. Thus we may assume $0<\alpha(f)<\|f\|$.

For $0<\delta\leqslant\|f\|$, we define the function $g_\delta\colon T\to X$ to be

$$g_\delta(t)=\begin{cases}\dfrac{f(t)}{\|f(t)\|}\,(\|f(t)\|-\delta) & \text{if}\quad \|f(t)\|\geqslant\delta\\ 0 & \text{if}\quad \|f(t)\|<\delta.\end{cases}$$

Obviously $g_\delta$ is continuous. Note also that $\|f-g_\delta\|=\delta$ and $\|g_\delta\|=\|f\|-\delta$. For $\alpha(f)<\delta\leqslant\|f\|$, by Theorem 2.2, there is $e\colon T\to S(X)$ continuous such that

$$e(t)=\frac{f(t)}{\|f(t)\|}\qquad \text{if}\quad \|f(t)\|\geqslant\delta.$$

Fix $\varepsilon>0$ and define $h\colon T\to X$ by

$$h(t)=g_\delta(t)+\varepsilon e(t),\qquad \forall t\in T.$$

It is easily seen that $h$ is in $Y^{-1}$ and that $\|h-g_\delta\|=\varepsilon$. Hence we conclude that $g_\delta\in\overline{Y^{-1}}$. With $\alpha=\alpha(f)$, it is immediate that $\|g_\delta-g_\alpha\|\leqslant\delta-\alpha$, hence $g_\alpha$ is in $\overline{Y^{-1}}$ and by the above $\|g_\alpha-f\|=\alpha=\alpha(f)$ and $\|g_\alpha\|=\|f\|-\alpha(f)$. Taking $g=g_\alpha$, the corollary follows. ∎

Proposition 2.5. *Let $T$ be a topological space and $X$ a normed space. Take $f$ in $Y$ and $\delta>0$. Assume that there exists a continuous mapping $e\colon T\to S(X)$ such that*

$$e(t)=\frac{f(t)}{\|f(t)\|}\qquad \textit{if}\quad \|f(t)\|\geqslant\delta.$$

*Then $\|f-e\|\leqslant\max\{\delta+1,\ \|f\|-1\}$.*

*Proof.* If $\|f(t)\|<\delta$, it follows that

$$\|f(t)-e(t)\|\leqslant\|f(t)\|+\|e(t)\|<\delta+1\leqslant\max\{\delta+1,\ \|f\|-1\}.$$

If $\|f(t)\|\geqslant\delta$, we have that

$$\|f(t)-e(t)\|=\left\|f(t)-\frac{f(t)}{\|f(t)\|}\right\|=|\,\|f(t)\|-1|.$$

When $\|f(t)\|\geqslant 1$, then

$$\|f(t)-e(t)\|=\|f(t)\|-1\leqslant\|f\|-1$$

and if $\|f(t)\| < 1$, we have

$$\|f(t) - e(t)\| = 1 - \|f(t)\| \leqslant 1 - \delta < 1 + \delta.$$

Hence if $\|f(t)\| \geqslant \delta$ then $\|f(t) - e(t)\| \leqslant \max\{\delta + 1, \|f\| - 1\}$.

So $\|f(t) - e(t)\| \leqslant \max\{\delta + 1, \|f\| - 1\}$ for every $t$ in $T$ and the proposition follows. ∎

Given a topological space $T$ and a strictly convex normed space $X$, it is known that $E(Y)$ consists of those $f \in Y$ such that $f(T)$ is contained in the unit sphere of $X$. In this case, we can determine the distance from an element $f$ in $Y$ to the set of extreme points of $B(Y)$.

THEOREM 2.6. *Let $T$ be a topological space and $X$ a strictly convex normed space. Consider an element $f$ in $Y$. Then*

$$\operatorname{dist}(f, E(Y)) = \begin{cases} \max\{1 - m(f), \|f\| - 1\} & \text{if } f \in Y^{-1} \\ \max\{1 + \alpha(f), \|f\| - 1\} & \text{if } f \notin Y^{-1}. \end{cases}$$

*Proof.* For any $f \in Y$, we always have

$$\|f - e\| \geqslant \|f\| - \|e\| = \|f\| - 1, \qquad \forall e \in E(Y).$$

Since $m(f) = m(e - (e - f)) \geqslant 1 - \|e - f\|$, $\forall e \in E(Y)$ it follows that $\|f - e\| \geqslant 1 - m(f)$, $\forall e \in E(Y)$ and so

$$\operatorname{dist}(f, E(Y)) \geqslant \max\{1 - m(f), \|f\| - 1\}$$

which verifies in particular if $f \in Y^{-1}$.

On the other hand, if $f \in Y^{-1}$, consider the mapping $e$ from $T$ into $S(X)$ given by $e(t) = f(t)/\|f(t)\|$ for each $t$ in $T$. A slight change in the proof of the last proposition shows that

$$\|f(t) - e(t)\| \leqslant \max\{1 - m(f), \|f\| - 1\} \qquad \forall t \in T,$$

and so $\operatorname{dist}(f, E(Y)) \leqslant \|f - e\| \leqslant \max\{1 - m(f), \|f\| - 1\}$, which proves the equality in case $f \in Y^{-1}$ and shows that $e$ is a best approximation to $f$ in $E(Y)$.

Let us now suppose that $f \notin Y^{-1}$. Given $\varepsilon \in \mathbb{R}^+$ and $e \in E(Y)$, define $\beta = \|f - e\| + \varepsilon$. It is clear that $\beta > 1$. Then we calculate

$$\begin{aligned} \|(\beta - 1)\, e(t) + f(t)\| &\geqslant \beta - \|e(t) - f(t)\| \\ &\geqslant \beta - \|e - f\| = \varepsilon > 0, \qquad \forall t \in T, \end{aligned}$$

whence $(\beta-1)\,e+f\in Y^{-1}$. But this implies that $\alpha(f)\leqslant\|f-e\|+\varepsilon-1$, hence $1+\alpha(f)\leqslant\|f-e\|$ because $\varepsilon$ is arbitrary. Since this holds for every $e$ in $E(Y)$, we have thus established the inequality

$$\operatorname{dist}(f, E(Y))\geqslant\max\{1+\alpha(f), \|f\|-1\}.$$

To prove the reverse inequality we observe the following

$$\operatorname{dist}(g, E(Y))\leqslant\max\{1, \|g\|-1\} \qquad \forall g\in Y^{-1},$$

which is due to the first part. By continuity

$$\operatorname{dist}(g, E(Y))\leqslant\max\{1, \|g\|-1\} \qquad \forall g\in \overline{Y^{-1}}.$$

By Corollary 2.4 there is a best approximation $g$ in $\overline{Y^{-1}}$ satisfying $\|f-g\|=\alpha(f)$ and $\|g\|=\|f\|-\alpha(f)$. Then

$$\begin{aligned}\operatorname{dist}(f, E(Y))&\leqslant\|f-g\|+\operatorname{dist}(g, E(Y))\\&\leqslant\alpha(f)+\max\{1, \|g\|-1\}\\&\leqslant\max\{1+\alpha(f), \|f\|-1\}. \quad \blacksquare\end{aligned}$$

A similar formula for C*-algebras was proved in [9, Theorem 2.7; 8, Theorem 10; 3, Theorem 2.3].

From the first part of the proof of Theorem 2.6, we obtain

COROLLARY 2.7. *Let $T$ be a topological space and $X$ a strictly convex normed space. Then $E(Y)$ is proximinal in $Y^{-1}$.*

With the hypotheses on $T$ and $X$ as in the above corollary, Example 14 in [8] shows that in this type of spaces $C(T, X)$ we can not expect in general to have best approximations in $E(Y)$ for functions $f$ with $\alpha(f)>0$. However, Theorem 2.2 permits us to get the following result.

PROPOSITION 2.8. *Let $T$ be a topological space and $X$ a strictly convex normed space. If $f$ is an element in $Y$ with $\|f\|>\alpha(f)+2$, then $f$ admits a best approximation in $E(Y)$.*

*Proof.* Let $\delta$ be given by $\|f\|-2>\delta>\alpha(f)$. By Theorem 2.2 we can then find $e\colon T\to S(X)$ continuous such that $e(t)=f(t)/\|f(t)\|$ if $\|f(t)\|\geqslant\delta$. Proposition 2.5 shows that $\|f-e\|\leqslant\max\{\delta+1, \|f\|-1\}=\|f\|-1$. Theorem 2.6 implies that

$$\operatorname{dist}(f, E(Y))=\max\{\alpha(f)+1, \|f\|-1\}=\|f\|-1$$

and we have the equality $\operatorname{dist}(f, E(Y))=\|f-e\|$. $\blacksquare$

The characterization of $\alpha(f)$, given in Corollary 2.3, involves a infimum. By [8, Example 14] this infimum can not be attained in general. The next result shows that if ($X$ is strictly convex and) $\alpha(f)$ is attained so is the distance $\operatorname{dist}(f, E(Y))$.

COROLLARY 2.9. *Let $T$ be a topological space and $X$ a strictly convex normed space. Let $f$ be in $Y$ with $\alpha(f) > 0$ and assume that there exists a continuous mapping $e: T \to S(X)$ such that*

$$e(t) = \frac{f(t)}{\|f(t)\|} \qquad \text{if} \quad \|f(t)\| \geqslant \alpha(f).$$

*Then $e$ is a best approximation for $f$ in $E(Y)$.*

*Proof.* The result follows immediately from Proposition 2.5 and Theorem 2.6. ∎

According to the above, the existence of best approximations in $E(Y)$ is closely related to the possibility of decomposing $f \in Y$ in the form

$$f(t) = \|f(t)\|\, e(t), \qquad \forall t \in T$$

for some continuous function $e: T \to S(X)$. This motivates the following.

DEFINITION 2.10. Let $T$ be a topological space and $X$ a normed space. It is said that a continuous function $f$ from $T$ into $X$ admits *a weak polar decomposition* if $f(t) = \|f(t)\|\, e(t)$ $\forall t \in T$ for some continuous mapping $e: T \to B(X)$. If a decomposition exists for every element in $C(T, X)$ we say that $C(T, X)$ has the *weak polar decomposition property*. Similarly we shall say that $C(T, X)$ has the *polar decomposition property* if, for every $f$ in $C(T, X)$, there is a continuous function $e: T \to S(X)$ such that $f(t) = \|f(t)\|\, e(t)$ $\forall t \in T$.

We proceed to study what spaces $C(T, X)$ have this property. For it we need remember the following concepts.

DEFINITION 2.11. Let $T$ be a topological space and $A$ a subset of $T$. We shall say that $A$ is a *cozero-set* if there is a continuous mapping $\varphi: T \to \mathbb{R}$ such that

$$A = \{t \in T: \varphi(t) \neq 0\}.$$

Two subsets $B_1$ and $B_2$ of $A$ are said to be *completely separated in $A$* if there exists a continuous function $\psi: A \to \mathbb{R}$ such that

$$\psi(t) = 0 \quad \forall t \in B; \qquad \psi(t) = 1 \quad \forall t \in B_2.$$

If $X$ is a real normed space we say that $A$ is $C(T, X)$-*embedded in* $T$ if every function in $C(A, X)$ admits an extension in $C(T, X)$. For $X = \mathbb{R}$, it is said that $A$ is $C^*$-*embedded in* $T$.

We will say that $T$ is an *F-space* if every cozero-set is $C^*$-embedded in $T$.

The basic result about $C^*$-embedding is Urysohn's extension theorem.

THEOREM 2.12 [6, 1.17]. *Let $T$ be a topological space and $A$ a subset of $T$. Then $A$ is $C^*$-embedded in $T$ if and only if any two completely separated sets in $A$ are completely separated in $T$.*

As a consequence the following is obtained.

COROLLARY 2.13 [6, 4.25]. *Let $T$ be a topological space. The following are equivalent conditions*:

(1) *$T$ is an F-space*

(2) *Every continuous function $f$: $T \to \mathbb{R}$ admits a weak polar decomposition.*

The result which follows can be proved easily.

PROPOSITION 2.14. *Let $T$ be a topological space and $X$ a normed space.*

(1) *If $X'$ is a normed space isomorphic to $X$, $C(T, X)$ has the polar decomposition property if and only if $C(T, X')$ has it.*

(2) *If $C(T, X)$ has the polar decomposition property, then so has every continuous function from $T$ into $X$ (not necessarily bounded).*

We now see that $\mathbb{R}$ may be replaced in the definition of $F$-spaces by any finite-dimensional real normed space.

PROPOSITION 2.15. *Let $T$ be a topological space. The following affirmations are equivalent.*

(1) *$T$ is an F-space.*

(2) *Every cozero-set is $C(T, X)$-embedded for each real normed space $X$ with finite dimension.*

*Proof.* *Necessity.* First suppose that $X = \mathbb{R}^n$. Let $A$ be a cozero-set and $f$: $A \to \mathbb{R}^n$ a continuous and bounded function. Then

$$f(t) = (f_1(t), ..., f_n(t)) \qquad \forall t \in A$$

and the functions $f_1, ..., f_n$ belong to $C(A, \mathbb{R})$.

By hypothesis, there are functions $\overline{f_1}, ..., \overline{f_n}$ in $C(T, \mathbb{R})$ such that

$$\overline{f_i}|_A = f_i, \qquad \forall i = 1, ..., n.$$

It is clear that $\bar{f} = (\overline{f_1}, ..., \overline{f_n})\colon T \to \mathbb{R}^n$ is a continuous and bounded extension of $f$.

In the general case, let $X$ be a real normed space and let $n = \dim X$. Let $H$ be an isomorphism from $X$ onto $\mathbb{R}^n$.

If $A$ is a cozero-set and $f \in C(A, X)$, it is clear that $H \circ f \in C(A, \mathbb{R}^n)$. By the above there exists $g \in C(T, \mathbb{R}^n)$ such that $g|_A = H \circ f$. Then $H^{-1} \circ g \in C(T, X)$. Moreover if $t \in A$, we have that $H^{-1} \circ g(t) = H^{-1}(H \circ g(t)) = f(t)$.

*Sufficiency*. It is sufficient to apply the hypothesis to $X = \mathbb{R}$; ∎

In [2, Theorem 2] it was proved that the $\lambda$-property in $C(T, X)$ with $X$ strictly convex is equivalent to the next property:

Let $T$ be a topological space and $X$ a normed space. The pair $(T, X)$ is said to have the *extension property* if the following holds: If $f\colon A \to S(X)$ is a continuous function from a closed subset $A$ of $T$ which has a continuous extension $g\colon T \to B(X)$ (i.e., $g|_A = f$), then $f$ actually has a continuous extension $e\colon T \to S(X)$.

Finally the next result provides us examples of pairs $(T, X)$ such that $C(T, X)$ has the polar decomposition property.

THEOREM 2.16. *Let $T$ be a topological space and $X$ a normed space with finite dimension. Then the following statements are equivalent*:

(1) *$T$ is an F-space and $(T, X)$ has the extension property.*

(2) *$C(T, X)$ has the polar decomposition property.*

*Proof.* (1) ⇒ (2). Let $f \in Y$ and define $g$ on $A = \{t \in T\colon \|f(t)\| > 0\}$ by $g(t) = f(t)/\|f(t)\|$. Since $A$ is a cozero-set, the continuous function $g\colon A \to S(X)$ admits an extension $\bar{g}$ in $C(T, X)$ by Proposition 2.15. By the continuity of $\bar{g}$ we have

$$\bar{g}(\bar{A}) \subset \overline{\bar{g}(A)} = \overline{g(A)} \subset S(X).$$

Then $\bar{g}|_{\bar{A}}$ is a continuous function from $\bar{A}$ into $S(X)$ such that its restriction to the set $A$ is $g$. The function $r \circ \bar{g}\colon T \to B(X)$ where $r\colon X \to B(X)$ is the mapping defined by

$$r(x) = \begin{cases} \dfrac{x}{\|x\|} & \text{if } \|x\| \geqslant 1 \\ x & \text{if } \|x\| \leqslant 1 \end{cases}$$

is a continuous extension of $\bar{g}|_{\bar{A}}$. Since $(T, X)$ has the extension property there exists $e\colon T \to S(X)$ continuous such that $e|_{\bar{A}} = \bar{g}|_{\bar{A}}$.

It is immediate to check that $f(t) = \|f(t)\|\, e(t)\ \forall t \in T$.

(2) ⇒ (1). Let $A$ be a closed subset of $T$ and $g\colon A \to S(X)$ a continuous function such that there exists $f\colon T \to B(X)$ continuous with $f|_A = g$.

By hypothesis, there is a $e\colon T \to S(X)$ continuous such that

$$f(t) = \|f(t)\|\, e(t) \qquad \forall t \in T.$$

If $t \in A$, we have

$$g(t) = f(t) = \|f(t)\|\, e(t) = e(t).$$

Hence $e$ is a continuous extension from $T$ into $S(X)$ for $g$ and therefore $(T, X)$ has the extension property.

We now show that $T$ is an $F$-space. By (2) every element of $C(T, X)$ has a polar decomposition. This also holds for every continuous function (not necessarily bounded) from $T$ into $\mathbb{R}^n$, where $n = \dim X$ by Proposition 2.14.

Let $f\colon T \to \mathbb{R}$ be a continuous function and $z = (z_1, ..., z_n) \in S(\mathbb{R}^n)$. The function $g\colon T \to \mathbb{R}^n$ given by

$$g(t) = f(t)\, z \qquad \forall t \in T$$

is continuous. Hence there exists a continuous function $e\colon T \to S(\mathbb{R}^n)$ such that

$$f(t)\, z = |f(t)|\, e(t) \qquad \forall t \in T.$$

Let $z_i \neq 0$. Then

$$f(t) = |f(t)|\, r\left(\frac{e_i(t)}{z_i}\right) \qquad \forall t \in T,$$

where $r\colon \mathbb{R} \to [-1, 1]$ is the natural retraction.

From the above it follows that every continuous function from $T$ into $\mathbb{R}$ admits a weak polar decomposition and, consequently, $T$ is an $F$-space by Corollary 2.13. ∎

The preceding result was obtained for $X = \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) in [4, Lemma 6].

Given a topological space $T$ we will denote by $\dim T$ the covering dimension of $T$ (see [5] for definitions). By using [10, Theorem $9_t$], Theorem 2.16 yields the following result.

COROLLARY 2.17. *Let $T$ be a completely regular F-space and $X$ a finite-dimensional strictly convex normed space. Suppose that* $\dim T < \dim X$. *Then $E(Y)$ is proximinal in $Y$.*

## 3. FUNCTIONS HAVING NO ZEROS AND $\lambda$-PROPERTY IN $C(T, X)$

If $Y$ is a normed space, consider for each $y$ in $B(Y)$ the possible convex combinations $y = \lambda e + (1-\lambda)\, g$, where $e \in E(Y)$ and $g \in B(Y)$. The supremum of all $\lambda$'s in such decompositions is denoted by $\lambda(y)$ by Aron and Lohman, and this defines the $\lambda$-function $\lambda$: $B(Y) \to [0, 1]$, see [1].

The space $Y$ is said to have the *$\lambda$ − property* if $\lambda(y) > 0$ for all $y$ in $B(Y)$, and $Y$ has the *uniform $\lambda$-property* if $Y$ verifies the $\lambda$-property and, in addition, satisfies

$$\inf\{\lambda(y)\colon y \in B(Y)\} > 0.$$

In [2] several characterizations of the $\lambda$-property in function spaces $C(T, X)$, with $T$ a topological space and $X$ a strictly convex normed space, were obtained. Moreover, they got a formula for the $\lambda$-function by assuming that $C(T, X)$ has the $\lambda$-property. A more complete theory may now be obtained with the aid of the functions in $C(T, X)$ which have no zeros.

In this section we will obtain the general expression of the $\lambda$-function in spaces $C(T, X)$ (with $X$ strictly convex). Our expression is also valid when $C(T, X)$ fails the $\lambda$-property.

Now we are ready to obtain the expression of the $\lambda$-function in $C(T, X)$.

THEOREM 3.1. *Let $T$ be a topological space and $X$ a strictly convex normed space. The $\lambda$-function on the unit ball $B(Y)$ for $Y = C(T, X)$ is given by the formula*

$$\lambda(f) = 1 - \tfrac{1}{2}\operatorname{dist}(f, E(Y)) = \begin{cases} \tfrac{1}{2}(1 + m(f)) & \text{if } f \in Y^{-1} \\ \tfrac{1}{2}(1 - \alpha(f)) & \text{if } f \notin Y^{-1} \end{cases}.$$

*Proof.* The second equality follows from Theorem 2.6. Let $f$ be in $B(Y)$ and $d = \operatorname{dist}(f, E(Y))$. If $f = \lambda e + (1-\lambda)\, g$ with $e \in E(Y)$ and $g \in B(Y)$, then it is clear that $d \leqslant \|f - e\| \leqslant 2(1-\lambda)$ and so $\lambda \leqslant 1 - (d/2)$. Since this holds for all decompositions, we conclude that $\lambda(f) \leqslant 1 - (d/2)$.

To prove the reverse inequality, first suppose that $f \in Y^{-1}$. Then $\alpha(f) = 0$ and $d = 1 - m(f)$. Let $e$ be in $E(Y)$ defined by $e(t) = f(t)/\|f(t)\|$ for every $t$ in $T$. Define $\lambda = 1 - (d/2) = \tfrac{1}{2}(1 + m(f))$. If $\lambda = 1$, then $\|f(t)\| = 1\ \forall t \in T$ and

$f = e$. Hence $\lambda(f) = 1 = 1 - (d/2)$. For $\lambda < 1$, let us define $g = (f - \lambda e)/(1 - \lambda)$. Since

$$\|g(t)\| = \frac{1}{1-\lambda} \left\| f(t) - \lambda \frac{f(t)}{\|f(t)\|} \right\| = \frac{|\,\|f(t)\| - \lambda|}{1-\lambda} \leqslant 1, \qquad \forall t \in T$$

$g \in B(Y)$. The expression $f = \lambda e + (1 - \lambda)\, g$ shows that

$$\lambda(f) \geqslant \lambda - 1 - \frac{d}{2}.$$

Let us now suppose that $f \notin Y^{-1}$. Then $m(f) = 0$ and $d = 1 + \alpha(f)$. Let $\lambda$ be $]0, 1 - (d/2)[$. Then $\alpha(f) < 1 - 2\lambda$. By Theorem 2.2, there exists a continuous function $e$ from $T$ into $S(X)$ such that

$$e(t) = \frac{f(t)}{\|f(t)\|} \qquad \text{if} \quad \|f(t)\| \geqslant 1 - 2\lambda.$$

Taking $g = (f - \lambda e)/(1 - \lambda)$, this reads $f = \lambda e + (1 - \lambda)\, g$ with $e \in E(Y)$ and $g \in B(Y)$. So $\lambda(f) \geqslant \lambda$. It follows that $\lambda(f) \geqslant 1 - (d/2)$, giving the desired equation. ∎

A similar result for $C^*$-algebras was proved by Brown and Pedersen in [3, Theorem 3.7].

Theorem 3.1 provides information about the problem of minimal decompositions of elements as convex combinations of extreme points.

COROLLARY 3.2. *Under the hypotheses of above theorem, if $f \in B(Y)$ and $f$ is $\lambda_1 e_1 + \cdots + \lambda_n e_n$ for $\lambda_1, ..., \lambda_n \in [0, 1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $e_1, ..., e_n$ in $E(Y)$, then $\lambda(f) > 0$ and*

$$n \geqslant \frac{1}{\lambda(f)} = \begin{cases} \dfrac{2}{1 + m(f)} & \text{if} \quad f \in Y^{-1} \\ \dfrac{2}{1 - \alpha(f)} & \text{if} \quad f \notin Y^{-1}. \end{cases}$$

*Proof.* It is clear that $\lambda_k \leqslant \lambda(f)$ for $k = 1, ..., n$ so, we get $1 \leqslant n\lambda(f)$. The proof is finished by Theorem 3.1. ∎

The next result shows the $\lambda$-function measures how close $f \in B(Y)$ is to being an extreme point of the unit ball.

COROLLARY 3.3. *Let $T$ be a topological space and $X$ a strictly convex normed space. Every $f$ in $B(Y)$ such that $\lambda(f)$ is attained admits a best approximation in $E(Y)$.*

*Proof.* Let $f$ be in $B(Y)$ such that $\lambda(f)$ is attained. Hence there are $e \in E(Y)$ and $g \in B(Y)$ such that $f = \lambda(f)e + (1-\lambda(f))\, g$. Then

$$\|f-e\| = (1-\lambda(f))\, \|g-e\| \leqslant 2(1-\lambda(f)) = \operatorname{dist}(f, E(Y))$$

by Theorem 3.1. Hence $e$ is a best approximation for $f$ in $E(Y)$. ∎

If $Y = C(T, X)$, it is clear that if the set $Y^{-1}$ is dense in $Y$, then $\alpha(f) = 0$ for every $f$ in $B(Y)$. The following proposition shows that, for general $C(T, X)$-spaces, $\sup \{\alpha(f) \colon f \in B(Y)\}$, is either 0 or 1.

PROPOSITION 3.4. *Let $T$ be a topological space, $X$ a normed space such that $Y^{-1}$ is nondense in $Y$. Then there is an $h$ in $Y$ with $\|h\| = \alpha(h) = 1$.*

*Proof.* We take $f$ in $Y$ with $\alpha = \alpha(f) > 0$ and define a continuous mapping $h \colon T \to X$ by

$$h(t) = \begin{cases} \dfrac{f(t)}{\|f(t)\|} & \text{if } \|f(t)\| \geqslant \alpha \\[2ex] \dfrac{f(t)}{\alpha} & \text{if } \|f(t)\| < \alpha. \end{cases}$$

Clearly $\|h\| = 1$. Then $\alpha(h) \leqslant 1$. Let us suppose to obtain a contradiction that $\alpha(h) < 1$. Let $\lambda$ be in $]\alpha(h), 1[$. By Theorem 2.2 the continuous function

$$t \mapsto \frac{h(t)}{\|h(t)\|} = \frac{f(t)}{\|f(t)\|}$$

defined for every $t$ with $\|h(t)\| \geqslant \lambda$ (that is, $\|f(t)\| \geqslant \alpha\lambda$), has a continuous extension $e \colon T \to S(X)$. By Corollary 2.3, $\alpha = \alpha(f) \leqslant \alpha\lambda < \alpha$, a contradiction. ∎

We now prove that the extension property can be reformulated by means of the functions which never vanish.

PROPOSITION 3.5. *Let $T$ be a topological space, $X$ a normed space. The following conditions are equivalent:*

(1) *The pair $(T, X)$ has the extension property.*

(2) *$Y^{-1}$ is dense in $Y$.*

*Proof.* (1) ⇒ (2). Let $h$ be in $Y$ and $\varepsilon > 0$. Let us define $g$: $T \to X$ by

$$g(t) = \begin{cases} \dfrac{h(t)}{\|h(t)\|} & \text{if} \quad \|h(t)\| \geqslant \dfrac{\varepsilon}{2} \\ \dfrac{2h(t)}{\varepsilon} & \text{if} \quad \|h(t)\| \leqslant \dfrac{\varepsilon}{2}. \end{cases}$$

It is clear that $g$ is continuous and $A = \{t \in T: \|h(t)\| = \varepsilon/2\}$ is closed. By hypothesis there exists $e$ from $T$ into $S(X)$ continuous such that $e(t) = h(t)/\|h(t)\|$ for $t$ in $A$. Let $y$ be in $Y$ defined by

$$y(t) = \begin{cases} h(t) & \text{if} \quad \|h(t)\| \geqslant \dfrac{\varepsilon}{2} \\ \dfrac{\varepsilon e(t)}{2} & \text{if} \quad \|h(t)\| \leqslant \dfrac{\varepsilon}{2}. \end{cases}$$

Evidently $y \in Y^{-1}$ and $\|h - y\| \leqslant \varepsilon$.

(2) ⇒ (1). Let us consider a continuous function $f$: $A \to S(X)$ defined on a closed set $A$ of $T$ such that there exists a continuous mapping $g$: $T \to B(X)$ with $g(t) = f(t)$ for $t$ in $A$. Since $\overline{Y^{-1}} = Y$, $\alpha(g) = 0$. By Theorem 2.2 there exists $e$: $T \to S(X)$ continuous such that

$$e(t) = \frac{g(t)}{\|g(t)\|} \qquad \text{if} \quad \|g(t)\| \geqslant 1.$$

It is clear that $e$ is an extension of $f$. ∎

Finally we give some new characterizations of the $\lambda$-property in spaces $Y = C(T, X)$ with $T$ a topological space and $X$ a strictly convex normed space by using the functions in $Y^{-1}$ (see also [2, 7]).

COROLLARY 3.6. *Let $T$ be a topological space and $X$ a strictly convex normed space. The following conditions are equivalent*:

(1) $\lambda(f) = \frac{1}{2}(1 + m(f))$ *for every $f$ in $B(Y)$.*

(2) *$Y$ has the uniform $\lambda$-property.*

(3) *$Y$ has the $\lambda$-property.*

(4) $\alpha(f) < 1$ *for every $f$ in $B(Y)$.*

(5) *$Y^{-1}$ is dense in $Y$.*

(6) *$(T, X)$ has the extension property.*

*Moreover, for $T$ completely regular and $X$ finite-dimensional, the conditions* (1)–(6) *are equivalent to*

(7) $\dim T < \dim X$.

*Proof.* The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial, $(3) \Rightarrow (4)$ and $(5) \Rightarrow (1)$ follow by Theorem 3.1, and $(4) \Rightarrow (5)$ is the Proposition 3.4. The Proposition 3.5 is $(5) \Leftrightarrow (6)$ and the equivalence $(6) \Leftrightarrow (7)$ is due essentially to Smyrnov [10, Theorem $9_t$]. ∎

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